

# Resonances, Unstable Systems and Irreversibility: Matter Meets Mind

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## Abstract

The fundamental time-reversal invariance of dynamical systems can be broken in various ways. One way is based on the presence of resonances and their interactions giving rise to unstable dynamical systems, leading to well-defined time arrows. Associated with these time arrows are semigroups bearing time orientations. Usually, when time symmetry is broken, two time-oriented semigroups result, one directed toward the future and one directed toward the past. If time-reversed states and evolutions are excluded due to resonances, then the status of these states and their associated backwards-in-time oriented semigroups is open to question. One possible role for these latter states and semigroups is as an abstract representation of mental systems as opposed to material systems. The beginnings of this interpretation will be sketched.

## 1 Introduction

Usually dynamical systems are considered to be time-reversible as their equations of motion are time-reversal symmetric under the time inversion operator  $R: (\vec{x}, t) \rightarrow (\vec{x}, -t)$ . This means that if  $\phi(t)$  is a solution of the equations of motion, then so is  $R\phi(t)$ . Such systems should then be reversible in the sense that if they exhibit a temporal succession of state transitions  $\phi_1, \phi_2, \phi_3, \dots, \phi_n$ , they can also exhibit the reverse temporal sequence  $R\phi_n, R\phi_{n-1}, R\phi_{n-2}, \dots, R\phi_1$ . In quantum mechanics these evolutions typically are described by one-parameter unitary groups of operators.

Resonances appear in a number of dynamical systems, both classical and quantum (e.g., Antoniou and Prigogine 1993; Antoniou and Tasaki 1993; Bohm et al. 1997) and are prototypical irreversible processes (e.g. scattering resonances). When the number of resonances in dynamical systems is sufficiently large, the dynamics is extremely unstable (e.g., exhibiting sensitive dependence on initial conditions), and becomes irreversible. For such unstable systems time

arrows for the dynamics can be clearly defined (e.g., Bishop 2004a; Bishop 2004b; Bishop 2005a). In the rigged Hilbert space framework for quantum mechanics (e.g., Bohm and Gadella 1989; Bohm et al. 1997), such time arrows are represented by semigroups. It is typically the case that there are two possible semigroups for such dynamics, one defined in the forward direction in time and one defined in the backward direction. However, if the evolutions of such resonance phenomena as scattering resonances and quasistable particles are irreversible, then there appears to be no physical relevance to the mathematical descriptions of the time-reversed states and evolutions.

After presenting the background of the rigged Hilbert space (RHS) framework for quantum mechanics (QM) in section 2, I will review its application to resonance states for scattering (section 3). This will be followed by a brief review of the extended Galilean group of Wigner and its application to resonance states in the RHS framework (section 4). I will then give an interpretation of the time-reversed resonance states and evolutions as abstract representations of mental systems as opposed to material systems (section 5).

## 2 Rigged Hilbert Space Quantum Mechanics

An RHS may be briefly characterized as follows. Let  $\Psi$  be an abstract linear scalar product space and complete  $\Psi$  with respect to two topologies. The first topology is the standard Hilbert space (HS) topology  $\tau_{\mathcal{H}}$  defined by the norm

$$\|h\| = \sqrt{(h, h)} \quad (1)$$

where  $h$  is an element of  $\Psi$ . The second topology  $\tau_{\Phi}$  is defined by a countable set of norms

$$\|\phi\|_n = \sqrt{(\phi, \phi)_n}, \quad n = 0, 1, 2, \dots \quad (2)$$

where  $\phi$  is also an element of  $\Psi$  and the scalar product in (2) is given by

$$(\phi, \phi')_n = (\phi, (\Delta + 1)^n \phi'), \quad n = 0, 1, 2, \dots \quad (3)$$

where  $\Delta$  is the Nelson operator  $\Delta = \sum_i \chi_i^2$ . The  $\chi_i$  are the generators of an enveloping algebra of observables for the system in question and they form a basis for a Lie algebra (Nelson 1959; Bohm et al. 1999). In the case of the harmonic oscillator, for example, the  $\chi_i$  would be the position and momentum operators or, alternatively, the raising and lowering operators. Furthermore if the operator  $\Delta + 1$  is *nuclear* then the space  $\Phi$  defined by (2) is a nuclear space (Treves 1967).

A Gel'fand triplet is obtained by completing  $\Psi$  with respect to  $\tau_{\Phi}$  to obtain  $\Phi$  and with respect to  $\tau_{\mathcal{H}}$  to obtain  $\mathcal{H}$ . In addition there are the dual spaces of continuous linear functionals  $\Phi^{\times}$  and  $\mathcal{H}^{\times}$  respectively. Since  $\mathcal{H}$  is self dual, we obtain

$$\Phi \subset \mathcal{H} \subset \Phi^{\times} . \quad (4)$$

The Nelson operator fully determines the space  $\Phi$ . However, there are many inequivalent irreducible representations of an enveloping algebra of a group characterizing a physical system (e.g. Bohm et al. 1999). Therefore further restrictions may be required to obtain a realization for  $\Phi$ , e.g., due to the convergence properties desired for test functions in  $\Phi$ . In general one chooses the weakest topology such that the algebra of operators for the physical problem is continuous and  $\Phi$  is nuclear. The physical symmetries of the system play an important role in such choices (Bohm et al. 1999).

In RHS QM, the observables form an algebra on the entire space of physical states (including  $\Phi^\times$ , where Dirac kets reside), so a RHS contains observables with continuous or even complex eigenvalues, whereas a HS does not. This means that the basis vector expansion of eigenvectors (Dirac's spectral decomposition) can be given a rigorous foundation resulting in the nuclear spectral theorem:

$$|\phi\rangle = \sum_n |E_n\rangle(E_n|\varphi) + \int |E\rangle\langle E|\varphi\rangle d\mu(E). \quad (5)$$

Here the rounded bras and kets denote elements of  $\mathcal{H}$  and the summation in (5) represents the discrete part of the spectrum. The angular bras,  $\langle\varphi|$ , denote elements defined in  $\Phi$ , while the angular kets,  $|E\rangle$ , denote elements defined in  $\Phi^\times$ ; hence, the integral in (5) represents the continuous part of the spectrum.

### 3 States, Observables and Resonances in Scattering

A typical scattering experiment consists of an accelerator, which prepares a projectile in a particular state, a target and detectors. The total Hamiltonian modeling the interaction of the particle with the target is, therefore,  $H = H_o + V$ , where  $H_o$  represents the free particle Hamiltonian and  $V$  the potential in the interaction region. The vectors representing growing and decaying states are associated with the resonance poles of the analytically continued S-matrix (Lax and Phillips 1967).

Following the Bohm group, a time arrow emerges in scattering resonances through imposing the *preparation/registration* arrow of time (Bohm et al. 1994; Bishop 2004b). The key intuition behind this arrow is that no observable properties of a state can be measured unless the state has first been prepared. Following Ludwig (1983; 1985), an in-state of a particular quantum system (considered as an ensemble of individual systems such as elementary particles) is prepared by a preparation apparatus (considered macrophysical). The detector (considered macrophysical) registers so-called out-states of post-interaction particles. In-states are taken to be elements  $\phi \in \Phi_-$  and observables are taken to be elements  $\psi \in \Phi_+$ . (Resonance states, such as the Dirac, Lippman, Schwinger kets and Gamow vectors, are elements of  $\Phi_\pm^\times$ ). This leads to a distinction between prepared states, on the one hand, and observables, each described by a separate

RHS (Bohm and Gadella 1989; Bohm et al. 1997):

$$\Phi_- \subset \mathcal{H} \subset \Phi_-^\times \quad (6a)$$

$$\Phi_+ \subset \mathcal{H} \subset \Phi_+^\times, \quad (6b)$$

where  $\Phi_-$  is the Hardy space of the lower complex energy half-plane intersected with the Schwartz class functions and  $\Phi_+$  is the Hardy space of the upper complex energy half-plane intersected with the Schwartz class functions. As Bohm and Gadella (1989) demonstrate, some elements of the generalized eigenstates in  $\Phi_-^\times$  and  $\Phi_+^\times$  correspond to exponentially growing and decaying states respectively. The semigroups governing these states are<sup>1</sup>

$$\langle \phi | U^\times | Z_R^* \rangle = e^{-iE_R t} e^{\frac{\Gamma}{2} t} \langle \phi | Z_R^* \rangle \quad t \leq 0, \quad t : -\infty \rightarrow 0 \quad (7a)$$

$$\langle \psi | U^\times | Z_R \rangle = e^{-iE_R t} e^{-\frac{\Gamma}{2} t} \langle \psi | Z_R \rangle \quad t \geq 0, \quad t : 0 \rightarrow \infty, \quad (7b)$$

where  $E_R$  represents the total resonance energy,  $\Gamma$  represents the resonance width,  $Z_R$  represents the pole at  $E_R - i\frac{\Gamma}{2}$ ,  $Z_R^*$  represents the pole at  $E_R + i\frac{\Gamma}{2}$ ,  $|Z_R^*\rangle \in \Phi_-^\times$  represents a growing Gamow vector and  $|Z_R\rangle \in \Phi_+^\times$  represents a decaying Gamow vector. The  $t < 0$  semigroup is identified as future-directed along with  $|Z_R^*\rangle$  as a forming/growing state. The  $t > 0$  semigroup is identified as future-directed along with  $|Z_R\rangle$  as a decaying state.

## 4 Time-reversed States and Observables

Following Wigner (1964), the time-reversal operator,  $R(t)$ , is the HS representation of the physical spacetime transformation

$$R : (\vec{x}, t) \rightarrow (\vec{x}, -t). \quad (8)$$

Therefore,  $R$  is an element of a co-representation of the extended Galilei symmetry group (Cariñena and Santander 1981) for a nonrelativistic spacetime (extended Poincaré group for a relativistic spacetime). These representations must be unitary and linear except for  $R$ , which is antilinear.

Wigner originally derived the properties of  $R$  for the spacetime symmetry group extended by time inversions and studied the parity inversion operator  $\Sigma$  and the total inversion operator  $T$  in combination with  $R$  (Wigner 1964). The parity inversion operator is unitary so its phase can be chosen such that  $\Sigma^2 = I$  (the identity operator), while  $T$  and  $R$  are both anti-unitary, so that the associative law for group multiplication then dictates that  $R^2 = \varepsilon_R I$  and  $T^2 = \varepsilon_T I$ , where  $\varepsilon_R = \pm 1$  and  $\varepsilon_T = \pm 1$ . The phase of  $T$  can be chosen so that

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<sup>1</sup>If  $U(t)$  is a unitary operator on  $\mathcal{H}$  and  $\Phi \subset \mathcal{H} \subset \Phi^\times$ , then  $U^\dagger$  can be *extended* to  $\Phi^\times$  provided that (1)  $U$  leaves  $\Phi$  invariant and (2)  $U$  is continuous on  $\Phi$  with respect to the topology  $\tau_\Phi$ . The operator  $U^\times$  denotes the *extension* of the HS operator  $U^\dagger$  to  $\Phi^\times$  and is defined by  $\langle U\phi | F \rangle = \langle \phi | U^\times F \rangle$  for all  $\phi \in \Phi$  and  $F \in \Phi^\times$ . When the group operator  $U^\dagger$  is extended to  $\Phi^\times$ , continuity requirements force the operators  $U^\times$  to be semigroups defined only on the temporal half-domains (Bohm and Gadella 1989).

$T = \Sigma R$  (where the order of application of  $\Sigma$  and  $R$  is physically immaterial). The extension of the Galilei spacetime symmetry group is summarized in Table I.

$\varepsilon_R$	$\varepsilon_T$	$\Sigma$	$R$	$T$
$(-1)^{2j}$	$(-1)^{2j}$	1	$C$	$C$
$-(-1)^{2j}$	$(-1)^{2j}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$
$(-1)^{2j}$	$-(-1)^{2j}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}$
$-(-1)^{2j}$	$-(-1)^{2j}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & C \\ -C & 0 \end{pmatrix}$

Table I. Properties of the Galilei spacetime symmetry group.

The index  $j$  refers to the spin of the particle being considered while  $C$  is an operator whose  $(2j+1)$ -dimensional matrix has the elements  $c_{\mu,\nu} = (-1)^{j+\mu}\delta_{\mu,\nu}$ , where  $-j \leq \mu$  and  $\nu \leq j$ . In the first representation, where  $\varepsilon_R = \varepsilon_T = (-1)^{2j}$ , there are no changes to the underlying vector space. This is the typical case discussed in QM (and relativistic quantum field theory). The other three representations, however, exhibit a doubling of the vector spaces (note the block matrices in the last three columns of Table I). In order to track this space doubling, let the index  $r = 0, 1$  label the rows and columns of the matrices in Table I.

Although no quantum fields have been constructed for representations two and three of Table I (indeed they are highly problematic), Bohm and co-workers have constructed models for the fourth representation by applying  $R$  to the states and observables in (7) (Bohm 1995; Bohm and Wickramasekara 1997). First, consider the growing Gamow vectors for,  $\phi^{r=0,\times} \in \Phi_-^{r=0,\times}$ . Applying  $R$  yields

$$R\phi^{r=0,\times} = \psi^{r=1,\times} \in \Phi_+^{r=1,\times}. \quad (9)$$

Similarly for the decaying Gamow vectors,  $\psi^{r=0,\times} \in \Phi_+^{r=0,\times}$ , applying  $R$  yields

$$R\psi^{r=0,\times} = \phi^{r=1,\times} \in \Phi_-^{r=1,\times}. \quad (10)$$

The transformation properties of  $R$  may be summarized as  $R : \Phi_{\pm}^{r=0,\times} \rightarrow \Phi_{\mp}^{r=1,\times}$ . The temporal evolution of these time-reversed vectors is also given by semigroups. Identify  $r = 0$  with the scattering experiment as normally carried out in the laboratory and  $r = 1$  with the extended spacetime transformed situation ("time-reversed counterparts"). Then  $U^\times(t)\langle\phi, r=0|Z_R^*, r=0\rangle \in \Phi_-^{r=0,\times}$ , a growing Gamow vector representing a preparable state for  $t \leq 0$ , is transformed under  $R$  into  $U^\times(-t)\langle\psi, r=1|Z_R, r=1\rangle \in \Phi_+^{r=1,\times}$ , where

$$e^{iE_R t} e^{-\frac{\Gamma}{2}t} \langle\psi, r=1|Z_R, r=1\rangle \quad (11)$$

is restricted to the time domain  $t \geq 0$  by continuity requirements. In the case of  $|Z_R^*, r=0\rangle$ , time counts up from  $-\infty$  to 0; in contrast, for  $|Z_R, r=1\rangle$ , time counts down from  $\infty$  to 0, meaning that it represents a Gamow vector

that increases as  $t$  decreases. Similarly,  $U^\times(t)\langle\psi, r = 0|Z_R, r = 0\rangle \in \Phi_+^{r=0,\times}$ , a decaying Gamow vector representing observables for  $t \geq 0$ , is transformed under  $R$  into  $U^\times(-t)\langle\phi, r = 1|Z_R^*, r = 1\rangle \in \Phi_-^{r=1,\times}$ , where

$$e^{iE_R t} e^{\frac{\Gamma}{2}t} \langle\phi, r = 1|Z_R^*, r = 1\rangle \quad (12)$$

is restricted to the time domain  $t \leq 0$  by continuity requirements. In the case of  $|Z_R, r = 0\rangle$ , time counts up from 0 to  $\infty$ ; in contrast, for  $|Z_R^*, r = 1\rangle$ , time counts down from 0 to  $-\infty$ , meaning that it represents a Gamow vector that decays as  $-t$  increases.

## 5 Matter Meets Mind

Comparing eqs. (7) with (11) and (12), we can see that in the  $r = 0$  regime the association of prepared states with growing eigenvectors and of detected observables with decaying eigenvectors is quite natural. On the other hand, the  $r = 1$  regime has no natural association with physical phenomena (to apply the eigenstates in this regime “straightforwardly” within our framework would lead to identifying the growing eigenvectors with “prepared observables” and the decaying eigenvectors with “detected states,” counterintuitive to say the least).<sup>2</sup>

Suppose we consider an alternative interpretation of the states and observables of the  $r = 1$  regime as an abstract representation of mental rather than material systems. The semigroups in this regime carry vectors from the future to the past. This could be taken as an abstract representation of final causation, appropriate to teleological or goal-directed behavior. For example, suppose I have a particular vision of the kind of person I want to become, say a more humble person; or suppose I have a particular goal I want to achieve, say landing a top-flight permanent academic position. These would be examples of final causation at work in everyday decisions and actions.<sup>3</sup> Drawing on the analogy with final causation as a backwards-directed influence, an eigenvector growing in the backwards time direction might represent the formation (“preparation” or “excitation”) of such a goal or vision of the future. This could be taken as representing the building influence of the goal or vision of the future on the present decision. Similarly, an eigenvector decaying in the backwards time direction might represent the decision state (“registration” or “de-excitation”) resulting in concrete action toward the goal. It is plausible that decision states decay back to some kind of “ready state” after action is initiated so that a new

<sup>2</sup>The question of interpreting the time-reversed states and observables was first suggested to me as an interesting problem by Arno Bohm.

<sup>3</sup>It would be interesting, though difficult, to connect the framework proposed here with analyses of goal-directed behavior in cognitive psychology, and cognitive science more broadly, as the latter perspectives tend to transmute the apparent final-causal nature of everyday goal-directed behavior into mechanisms of efficient causation (e.g. Bishop 2005b). Hence, establishing the desired connection is not straightforward without at least extending the current cognitive paradigms.

decision state can be created for the next set of goals and actions. The rate of decay could be slower or faster depending on whether the intended action required more effort of will to “stay on track” as it were to completion or not. The resonance state might be taken as a representation of the decision itself.

The  $r = 1$  regime could, then, serve as an abstract model of goal-directed decision and action. Moreover, both regimes together would play a role in the abstract description of mental and material systems and their relations. The  $r = 0$  regime would correspond to material systems while the  $r = 1$  regime would correspond to mental systems. We would, then, have a unified abstract description of mental-material systems.

Such an abstract description could be deployed to represent a “dualistic” distinction between material and mental domains, emerging from a “monistic” domain without such a distinction. It has been proposed that this emergence is related to some temporal symmetry breaking (Atmanspacher 2003; Primas 2004) in the spirit of ideas of Pauli and Jung (Pauli and Enz 2001), where physical and psychical aspects originate in a psychophysically neutral domain. The symmetry breaking envisaged need not be a unique, one-time event, but is perhaps best understood as an ongoing process due to a number of contingent conditions giving rise to the mental-material distinction.<sup>4</sup> Furthermore, this symmetry breaking can lead to the Cartesian distinction of the dualistic approach while still allowing for correlations or forms of interaction emerging from the neutral domain, perhaps leading to resolution of a number of problems plaguing the dualistic approach.

To be a bit more precise, suppose the neutral domain is characterized by states  $\omega$  and a unitary symmetry (continuum order, automorphic dynamics, etc.). The dynamics of this domain would then exhibit the time-reversal symmetry described in section 1 and might be characterized by a one-parameter unitary group of bounded operators on  $\mathcal{H}$ . Some, as yet unspecified, symmetry breaking leads to the generation of time-asymmetric dynamics characterized by semigroups governing the two regimes,  $r = 0, 1$ . As originally characterized, the states  $\omega$  are neutral with respect to mental-material aspects, whereas after the symmetry breaking, the states are differentiated into material ( $r = 0$ ) and mental ( $r = 1$ ) states and processes. It is at the level of symmetry breaking that the states and observables discussed above emerge. The characterization of observables in the unitary domain is left unaddressed here.<sup>5</sup>

The fact that the  $r = 0$  and  $r = 1$  regimes are related to each other via a time-reversal operator suggests the possibility that there is some form of intertwining relation among the states and observables of the two regimes. If so, then the relationship among the elements of the mental and material domains would not be so starkly disjoint as in Descartés’ view, where the two domains are

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<sup>4</sup>The role of contingent conditions in the emergence of properties is discussed in (Bishop and Atmanspacher 2005).

<sup>5</sup>As a referee helpfully pointed out, one could also consider a more general kind of interpretation of the  $r = 1$  regime, namely as representing observational systems, where mental systems are a very important special case. Space does not permit consideration of this interesting possibility.

conceived as distinctly different kinds of substances. Therefore, the distinction between mental and material domains *need not* imply Descartés' metaphysical distinction nor the kinds of interaction problems encountered in that view.

The RHS framework for QM allows for the description of both time-symmetric and time-asymmetric phenomena. In particular, it is well-suited for the description of resonances and other kinds of unstable states. If the interpretation sketched here makes the time-reversed states and observables of the  $r = 1$  regime plausible, then the RHS framework is also well-suited for such an abstract representation of mental and material states. The unitary neutral domain might be related to  $\mathcal{H}$  while the  $r = 0$  (material) and  $r = 1$  (mental) regimes are related to  $\Phi$  and  $\Phi^\times$ .

Although one might wonder about the propriety of using concrete models to motivate the framework and then subsequently throwing those models to the side to apply the framework to more abstract questions, this way of proceeding represents a well-established use of models in mathematical physics (Redhead 1980). There are also technical questions about the application of Wigner's ideas to observables as well as to semigroup representations, but these questions are fairly straightforward. What is not so straightforward are questions such as the emergence of time, or the kinds of contingent conditions leading to the symmetry breaking generating the two regimes. The abstract RHS framework proposed here appears to be promising as one avenue for exploring such topics.

## 6 Concluding Summary

One way the fundamental time-reversal invariance of dynamical systems might be broken is through the presence of resonances and their interactions giving rise to unstable dynamical systems. When time-reversal invariance is broken, this results in two well-defined time arrows associated with semigroups bearing time orientations, one directed toward the future and one directed toward the past. Scattering resonances provide an example where time-reversal invariance is broken. The resulting forward-directed semigroups and states correspond to the processes of resonance formation, decay and detection, but the backward-directed semigroups and states are thought to have perhaps only mathematical significance. Here, I have sketched a possible interpretation of these latter semigroups and states as corresponding abstractly to the domain of mental systems, while the forward-directed semigroups and states would correspond to the domain of material systems. The crucial idea is that these two domains might emerge from a more fundamental domain that is neutral with respect to any mental-material distinction and that, hence, various possibilities exist for relations between the emergent domains that are typically precluded by traditional Cartesian dualisms. The RHS framework seems well-suited for describing and exploring these possibilities.

**Acknowledgement 1** *I would like to thank H. Atmanspacher and A. Bohm for helpful discussions. Financial support from the Alexander von Humboldt*



Foundation as well as the Federal Ministry of Education and Research and the German government's Program for the Investment in the Future of the are gratefully acknowledged.

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